EXERCISE 23

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1. Find $\vec{a}.\vec{b}$ when

(i)

$\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$ and $\vec{b} = 3\hat{i} - 4\hat{j} - 2\hat{k}$

(ii)

$\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = -2\hat{j} + 4\hat{k}$

(iii)

$\vec{a} = \hat{i} - \hat{j} + 5\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{k}$

Solution:

(i) It is given that

$$\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$$

$$\vec{b} = 3\hat{i} - 4\hat{j} - 2\hat{k}$$

We can write it as

$$\vec{a}.\vec{b} = (\hat{i} - 2\hat{j} + \hat{k}).(3\hat{i} - 4\hat{j} - 2\hat{k})$$

On further calculation

$$\vec{a}.\vec{b} = (1 \times 3) + (-2 \times -4) + (1 \times -2)$$

So we get

$$\vec{a}.\vec{b} = 3 + 8 - 2 = 9$$

(ii) It is given that

$$\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\vec{b} = 0\hat{i} - 2\hat{j} + 4\hat{k}$$

We can write it as

$$\vec{a}.\vec{b} = (\hat{i} + 2\hat{j} + 3\hat{k}).(0\hat{i} - 2\hat{j} + 4\hat{k})$$

On further calculation

$$\vec{a}.\vec{b} = (1 \times 0) + (2 \times -2) + (3 \times 4)$$

So we get

$$\vec{a}.\vec{b} = 0 - 4 + 12 = 8$$

(iii) It is given that

$\vec{a} = \hat{i} - \hat{j} + 5\hat{k}$

$\vec{b} = 3\hat{i} + 0\hat{j} - 2\hat{k}$

We can write it as

$\vec{a}.\vec{b} = (\hat{i} - \hat{j} + 5\hat{k}).(3\hat{i} + 0\hat{j} - 2\hat{k})$

On further calculation

$\vec{a}.\vec{b} = (1 \times 3) + (-1 \times 0) + (5 \times -2)$

So we get

$\vec{a}.\vec{b} = 3 - 0 - 10 = -7$

2. Find the value of λ for which $\vec{a}$ and $\vec{b}$ are perpendicular, where

(i)

$\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = (\hat{i} - 2\hat{j} + 3\hat{k})$

(ii)

$\vec{a} = 3\hat{i} - \hat{j} + 4\hat{k}$ and $\vec{b} = -\lambda\hat{i} + 3\hat{j} + 3\hat{k}$

(iii)

$\vec{a} = 2\hat{i} + 4\hat{j} - \hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \lambda\hat{k}$

(iv)

$\vec{a} = 3\hat{i} + 2\hat{j} - 5\hat{k}$ and $\vec{b} = -5\hat{j} + \lambda\hat{k}$

Solution:

(i) It is given that

$\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$

$\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$

We know that the two vectors are perpendicular then the dot product is zero.

Here

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta = |\vec{a}||\vec{b}|\cos\frac{\pi}{2} = 0$

By substituting the values

$$\vec{a}.\vec{b} = (2\hat{i} + \lambda\hat{j} + \hat{k}).(\hat{i} - 2\hat{j} + 3\hat{k}) = 0$$

It can be written as

$$\vec{a}.\vec{b} = (2 \times 1) + (\lambda \times -2) + (1 \times 3) = 0$$

On further simplification

$$\vec{a}.\vec{b} = 2 - 2\lambda + 3 = 0$$

So we get

$5 = 2\lambda$

$\lambda = 5/2$

(ii) It is given that

$$\vec{a} = 3\hat{i} - \hat{j} + 4\hat{k}$$

$$\vec{b} = -\lambda + 3\hat{j} + 3\hat{k}$$

We know that the two vectors are perpendicular then the dot product is zero.

Here

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta = |\vec{a}||\vec{b}|\cos\frac{\pi}{2} = 0$$

By substituting the values

$$\vec{a}.\vec{b} = (3\hat{i} - \hat{j} + 4\hat{k}).(-\lambda + 3\hat{j} + 3\hat{k}) = 0$$

It can be written as

$$\vec{a}.\vec{b} = (3 \times -\lambda) + (-1 \times 3) + (4 \times 3) = 0$$

On further simplification

$$\vec{a}.\vec{b} = -3\lambda - 3 + 12 = 0$$

So we get

$9 = 3\lambda$

$\lambda = 9/3 = 3$

(iii) It is given that

$$\vec{a} = 2\hat{i} + 4\hat{j} - \hat{k}$$

$\vec{b} = 3\hat{i} - 2\hat{j} + \lambda\hat{k}$

We know that the two vectors are perpendicular then the dot product is zero.

Here

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta = |\vec{a}||\vec{b}|\cos\frac{\pi}{2} = 0$$

By substituting the values

$$\vec{a}.\vec{b} = (2\hat{i} + 4\hat{j} - \hat{k}).(3\hat{i} - 2\hat{j} + \lambda\hat{k}) = 0$$

It can be written as

$$\vec{a}.\vec{b} = (2 \times 3) + (4 \times -2) + (-1 \times \lambda) = 0$$

On further simplification

$$\vec{a}.\vec{b} = -\lambda + 6 - 8 = 0$$

So we get

$-2 = \lambda$

$\lambda = -2$

(iv) It is given that

$\vec{a} = 3\hat{i} + 2\hat{j} - 5\hat{k}$

$\vec{b} = -5\hat{j} + \lambda\hat{k}$

We know that the two vectors are perpendicular then the dot product is zero.

Here

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta = |\vec{a}||\vec{b}|\cos\frac{\pi}{2} = 0$$

By substituting the values

$$\vec{a}.\vec{b} = (3\hat{i} + 2\hat{j} - 5\hat{k}).(-5\hat{j} + \lambda\hat{k}) = 0$$

It can be written as

$$\vec{a}.\vec{b} = (3 \times 0) + (2 \times -5) + (-5 \times \lambda) = 0$$

On further simplification

$$\vec{a}.\vec{b} = -5\lambda + 0 - 10 = 0$$

So we get

$-10 = 5\lambda$

$\lambda = -10/5 = -2$

3. (i) If $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$, show that $\left(\vec{a} + \vec{b}\right)$ is perpendicular to $\left(\vec{a} - \vec{b}\right)$.
(ii) If $\vec{a} = \left(5\hat{i} - \hat{j} - 3\hat{k}\right)$ and $\vec{b} = \left(\hat{i} + 3\hat{j} - 5\hat{k}\right)$ then show that $\left(\vec{a} + \vec{b}\right)$ and $\left(\vec{a} - \vec{b}\right)$ are orthogonal.
Solution:

(i) It is given that

$$\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$$

$$\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$$

We know that

$$\vec{a} + \vec{b} = \hat{i} + 2\hat{j} - 3\hat{k} + 3\hat{i} - \hat{j} + 2\hat{k}$$

On further calculation

$$\vec{a} + \vec{b} = 4\hat{i} + \hat{j} - \hat{k}$$

Similarly

$$\vec{a} - \vec{b} = \hat{i} + 2\hat{j} - 3\hat{k} - (3\hat{i} - \hat{j} + 2\hat{k})$$

On further calculation

$$\vec{a} - \vec{b} = -2\hat{i} + 3\hat{j} - 5\hat{k}$$

Here we can write it as

$$(\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = (4\hat{i} + \hat{j} - \hat{k}).(-2\hat{i} + 3\hat{j} - 5\hat{k})$$

So we get

$= (4 \times -2) + (1 \times 3) + (-1 \times -5)$

$= -8 + 3 + 5$

$= 0$

Hence, it is proved that the $\left(\vec{a} + \vec{b}\right)$ is perpendicular to $\left(\vec{a} - \vec{b}\right)$.

(ii) It is given that

$$\vec{a} = 5\hat{i} - \hat{j} - 3\hat{k}$$

$$\vec{b} = \hat{i} + 3\hat{j} - 5\hat{k}$$

We know that

$$\vec{a} + \vec{b} = 5\hat{i} - \hat{j} - 3\hat{k} + \hat{i} + 3\hat{j} - 5\hat{k}$$

On further calculation

$$\vec{a} + \vec{b} = 6\hat{i} + 2\hat{j} - 8\hat{k}$$

Similarly

$$\vec{a} - \vec{b} = 5\hat{i} - \hat{j} - 3\hat{k} - (\hat{i} + 3\hat{j} - 5\hat{k})$$

On further calculation

$$\vec{a} - \vec{b} = 4\hat{i} - 4\hat{j} + 2\hat{k}$$

Here we can write it as

$$(\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = (6\hat{i} + 2\hat{j} - 8\hat{k}).(4\hat{i} - 4\hat{j} + 2\hat{k})$$

So we get

$= (6 \times 4) + (2 \times - 4) + (- 8 \times 2)$

$= 24 - 8 - 16$

$= 0$

Here the dot product of $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ is zero.

Hence, it is proved that the $(\vec{a} + \vec{b})$ is orthogonal to $(\vec{a} - \vec{b})$.

4. If $\vec{a} = (\hat{i} - \hat{j} + 7\hat{k})$ and $\vec{b} = (5\hat{i} - \hat{j} + \lambda\hat{k})$ then find the value of λ so that $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ are orthogonal vectors.

Solution:

It is given that

$$\vec{a} = \hat{i} - \hat{j} + 7\hat{k}$$

$$\vec{b} = 5\hat{i} - \hat{j} + \lambda\hat{k}$$

We know that

$$(\vec{a} + \vec{b}) = \hat{i} - \hat{j} + 7\hat{k} + 5\hat{i} - \hat{j} + \lambda\hat{k}$$

On further calculation

$$\vec{a} + \vec{b} = 6\hat{i} - 2\hat{j} + (7 + \lambda)\hat{k}$$

Similarly

$$\vec{a} - \vec{b} = \hat{i} - \hat{j} + 7\hat{k} - (5\hat{i} - \hat{j} + \lambda\hat{k})$$

On further calculation

$$\vec{a} - \vec{b} = -4\hat{i} + 0\hat{j} + (7 - \lambda)\hat{k}$$

Here we can write it as

$$(\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = (6\hat{i} - 2\hat{j} + (7 + \lambda)\hat{k}).(-4\hat{i} + 0\hat{j} + (7 - \lambda)\hat{k})$$

So the product of two orthogonal vectors is zero

$(6 \times -4) + (-2 \times 0) + [(7 + \lambda) \times (7 - \lambda)] = 0$

$-24 + 0 + (49 - \lambda^2) = 0$

On further calculation

$\lambda^2 = 25$

We get

$\lambda = \pm 5$

5. Show that the vectors
$$\frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k}), \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k}) \text{ and } \frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k})$$
are mutually perpendicular unit vectors.

Solution:

Consider

$$\vec{a} = \frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k})$$

$$\vec{b} = \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k})$$

$$\vec{c} = \frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k})$$

We know that

$|\vec{a}| = |\vec{b}| = |\vec{c}| = 1$

To prove:

$\vec{a}.\vec{b} = \vec{b}.\vec{c} = \vec{a}.\vec{c} = 0$

Let us consider LHS

$$\vec{a}.\vec{b} = \frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k}).\frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k})$$

We get

$$= \frac{1}{49}(6 - 18 + 12) = 0$$

Similarly

$$\vec{b}.\vec{c} = \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k}).\frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k})$$

We get

$$= \frac{1}{49}(18 - 12 - 6) = 0$$

And

$$\vec{a}.\vec{c} = \frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k}).\frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k})$$

We get

$$= \frac{1}{49}(12 + 6 - 18) = 0$$

Here LHS = RHS

Therefore, it is proved that the vectors are mutually perpendicular unit vectors.

6. Let $\vec{a} = 4\hat{i} + 5\hat{j} - \hat{k}, \vec{b} = \hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} - \hat{k}$.

Find a vector $\vec{d}$ **which is perpendicular to both** $\vec{a}$ **and** $\vec{b}$**, and is such that** $\vec{d}.\vec{c} = 21$.

Solution:

It is given that

$\vec{a} = (4\hat{i} + 5\hat{j} - \hat{k})$

$\vec{b} = (\hat{i} - 4\hat{j} + 5\hat{k})$

$\vec{c} = (3\hat{i} + \hat{j} - \hat{k})$

Consider $\vec{d} = p\hat{i} + q\hat{j} + r\hat{k}$

Here $\vec{d}$ is perpendicular to both $\vec{a}$ and $\vec{b}$

$\vec{d}.\vec{a} = \vec{d}.\vec{b} = 0$

By substituting the values

$(p\hat{i} + q\hat{j} + r\hat{k}).(4\hat{i} + 5\hat{j} - \hat{k}) = 0$

We get

$4p + 5q - r = 0$ …… (i)

Similarly

$(p\hat{i} + q\hat{j} + r\hat{k}).(\hat{i} - 4\hat{j} + 5\hat{k}) = 0$

We get

$p - 4q + 5r = 0$ ……. (ii)

It is given that

$\vec{d} \cdot \vec{c} = 21$

Substituting the values

$(p\hat{i} + q\hat{j} + r\hat{k}).(3\hat{i} + \hat{j} - \hat{k}) = 21$

We get

$3p + q - r = 21$ ….. (iii)

By solving the equations simultaneously

$p = 7$, $q = -7$ and $r = -7$

It can be written as

$\vec{d} = p\hat{i} + q\hat{j} + r\hat{k} = 7\hat{i} - 7\hat{j} - 7\hat{k}$

By taking 7 as common

$\vec{d} = 7(\hat{i} - \hat{j} - \hat{k})$

7. Let $\vec{a} = (2\hat{i} + 3\hat{j} + 2\hat{k})$ **and** $\vec{b} = (\hat{i} + 2\hat{j} + \hat{k})$.
Find the projection of (i) $\vec{a}$ **on** $\vec{b}$ **and (ii)** $\vec{b}$ **on** $\vec{a}$.
Solution:

It is given that

$$\vec{a} = (2\hat{i} + 3\hat{j} + 2\hat{k})$$

$$\vec{b} = (\hat{i} + 2\hat{j} + \hat{k})$$

We know that

$$|\vec{a}| = \sqrt{2^2 + 3^2 + 2^2}$$

$$= \sqrt{4 + 9 + 4}$$

$$= \sqrt{17}$$

Similarly

$$|\vec{b}| = \sqrt{1^2 + 2^2 + 1^2}$$

$$= \sqrt{1 + 4 + 1}$$

$$= \sqrt{6}$$

It can be written as

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

$$= \frac{2\hat{i} + 3\hat{j} + 2\hat{k}}{\sqrt{17}}$$

Similarly

$$\hat{b} = \frac{\vec{b}}{|\vec{b}|}$$

$$= \frac{\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{6}}$$

So the projection of

$\vec{a}$ on $\vec{b}$ is $\vec{a}\hat{b} = (2\hat{i} + 3\hat{j} + 2\hat{k}).\frac{\hat{i}+2\hat{j}+\hat{k}}{\sqrt{6}}$

We get

$= \frac{2+6+2}{\sqrt{6}} = \frac{10}{\sqrt{6}} = \frac{5\sqrt{6}}{3}$

In the same way

$\vec{b}$ on $\vec{a}$ is $\vec{b}\hat{a} = (\hat{i} + 2\hat{j} + \hat{k}).\frac{2\hat{i}+3\hat{j}+2\hat{k}}{\sqrt{17}}$

We get

$= \frac{2+6+2}{\sqrt{17}} = \frac{10}{\sqrt{17}} = \frac{10\sqrt{17}}{17}$

8. Find the projection of $(8\hat{i} + \hat{j})$ in the direction of $(\hat{i} + 2\hat{j} - 2\hat{k})$.
Solution:

It is given that

$\vec{a} = (8\hat{i} + \hat{j})$

$\vec{b} = (\hat{i} + 2\hat{j} - 2\hat{k})$

We know that

$|\vec{b}| = \sqrt{1^2 + 2^2 + 2^2}$

$= \sqrt{1 + 4 + 4} = \sqrt{9} = 3$

It can be written as

$\hat{b} = \frac{\vec{b}}{|\vec{b}|} = \frac{\hat{i} + 2\hat{j} - 2\hat{k}}{3}$

So the projection of $(8\hat{i} + \hat{j})$ on $(\hat{i} + 2\hat{j} - 2\hat{k})$ can be written as

$(8\hat{i} + \hat{j}).\frac{\hat{i}+2\hat{j}-2\hat{k}}{3} = \frac{8+2+0}{3}$

We get

= 10/3

9. Write the projection of vector $(\hat{i} + \hat{j} + \hat{k})$ along the vector $\hat{j}$.
Solution:

It is given that

$$\vec{a} = (\hat{i} + \hat{j} + \hat{k})$$

$$\vec{b} = (\hat{j})$$

We know that

$$|\vec{b}| = \sqrt{0^2 + 1^2 + 0^2}$$

$$= \sqrt{1} = 1$$

It can be written as

$$\hat{b} = \frac{\vec{b}}{|\vec{b}|} = \frac{(\hat{j})}{1}$$

So the projection of $(\hat{i} + \hat{j} + \hat{k})$ on $\hat{j}$ can be written as

$$(\hat{i} + \hat{j} + \hat{k}).(\hat{j}) = 1$$

10. (i) Find the projection of $\vec{a}$ on $\vec{b}$ if $\vec{a} \cdot \vec{b} = 8$ and $\vec{b} = (2\hat{i} + 6\hat{j} + 3\hat{k})$.

(ii) Write the projection of the vector $(\hat{i} + \hat{j})$ on the vector $(\hat{i} - \hat{j})$.

Solution:

(i) It is given that

$$\vec{b} = (2\hat{i} + 6\hat{j} + 3\hat{k})$$

We can write it as

$$|\vec{b}| = \sqrt{2^2 + 6^2 + 3^2}$$

$$= \sqrt{4 + 36 + 9} = \sqrt{49} = 7$$

So the projection of $\vec{a}$ on $\vec{b}$

$$= \vec{a}.\frac{\vec{b}}{|\vec{b}|} = \frac{8}{7}$$

(ii) It is given that

$$\vec{a} = (\hat{i} + \hat{j})$$

$$\vec{b} = (\hat{i} - \hat{j})$$

We can write it as

$$|\vec{b}| = \sqrt{1^2 + (-1)^2}$$

$$= \sqrt{1 + 1} = \sqrt{2}$$

Here we get

$$\hat{b} = \frac{\vec{b}}{|\vec{b}|} = \frac{\hat{i} - \hat{j}}{\sqrt{2}}$$

So the projection of $(\hat{i} + \hat{j})$ on $(\hat{i} - \hat{j})$

$$(\hat{i} + \hat{j}).\frac{\hat{i}-\hat{j}}{\sqrt{2}} = \frac{1-1}{\sqrt{2}} = 0$$

11. Find the angle between the vectors $\vec{a}$ and $\vec{b}$, when
(i)
$\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$
(ii)
$\vec{a} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} - 2\hat{j} + 4\hat{k}$
(iii)
$\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = \hat{j} + \hat{k}$
Solution:

(i) It is given that

$$\vec{a} = (\hat{i} - 2\hat{j} + 3\hat{k})$$

$$\vec{b} = (3\hat{i} - 2\hat{j} + \hat{k})$$

It can be written as

$$|\vec{a}| = \sqrt{1^2 + (-2)^2 + 3^2}$$

$$= \sqrt{1 + 4 + 9} = \sqrt{14}$$

Similarly

$$|\vec{b}| = \sqrt{3^2 + (-2)^2 + 1^2}$$

$$= \sqrt{9 + 4 + 1} = \sqrt{14}$$

We know that

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

By substituting the values

$(\hat{i} - 2\hat{j} + 3\hat{k})(3\hat{i} - 2\hat{j} + \hat{k}) = \sqrt{14}\sqrt{14}\cos\theta$

On further calculation

$(3 + 4 + 3) = 14 \cos \theta$

It can be written as

$\cos \theta = 10/14 = 5/7$

So we get

$\theta = \cos^{-1} (5/7)$

(ii) It is given that

$\vec{a} = (3\hat{i} + \hat{j} + 2\hat{k})$

$\vec{b} = (2\hat{i} - 2\hat{j} + 4\hat{k})$

It can be written as

$|\vec{a}| = \sqrt{3^2 + (1)^2 + 2^2}$

$= \sqrt{9 + 1 + 4} = \sqrt{14}$

Similarly

$|\vec{b}| = \sqrt{2^2 + (-2)^2 + 4^2}$

$= \sqrt{4 + 4 + 16} = \sqrt{24}$

We know that

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

By substituting the values

$(3\hat{i} + \hat{j} + 2\hat{k})(2\hat{i} - 2\hat{j} + 4\hat{k}) = \sqrt{14}\sqrt{24}\cos\theta$

On further calculation

$(6 - 2 + 8) = \sqrt{336} \cos \theta$

It can be written as

$\cos\theta = 12/\sqrt{336} = \sqrt{(144/336)}$

So we get

$\theta = \cos^{-1}\sqrt{(3/7)}$

(iii) It is given that

$\vec{a} = (\hat{i} - \hat{j})$

$\vec{b} = (\hat{j} + \hat{k})$

It can be written as

$|\vec{a}| = \sqrt{1^2 + (-1)^2}$

$= \sqrt{1 + 1} = \sqrt{2}$

Similarly

$|\vec{b}| = \sqrt{(1)^2 + 1^2}$

$= \sqrt{1 + 1} = \sqrt{2}$

We know that

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

By substituting the values

$(\hat{i} - \hat{j})(\hat{j} + \hat{k}) = \sqrt{2}\sqrt{2}\cos\theta$

On further calculation

$(-1) = 2\cos\theta$

It can be written as

$\cos\theta = -1/2$

So we get

$\theta = \cos^{-1}(-1/2)$

$\theta = 120^\circ$

12. If $\vec{a} = (\hat{i} + 2\hat{j} - 3\hat{k})$ and $\vec{b} = (3\hat{i} - \hat{j} + 2\hat{k})$ then calculate the angle between $(2\vec{a} + \vec{b})$ and $(\vec{a} + 2\vec{b})$.

Solution:

It is given that

$\vec{a} = (\hat{i} + 2\hat{j} - 3\hat{k})$

$\vec{b} = (3\hat{i} - \hat{j} + 2\hat{k})$

We know that

$\vec{a} + 2\vec{b} = (\hat{i} + 2\hat{j} - 3\hat{k}) + 2(3\hat{i} - \hat{j} + 2\hat{k})$

$= 7\hat{i} + \hat{k}$

$2\vec{a} + \vec{b} = 2(\hat{i} + 2\hat{j} - 3\hat{k}) + (3\hat{i} - \hat{j} + 2\hat{k})$

$= 5\hat{i} + 3\hat{j} - 4\hat{k}$

So we get

$|\vec{a} + 2\vec{b}| = \sqrt{7^2 + (1)^2}$

$= \sqrt{49 + 1} = \sqrt{50}$

$|2\vec{a} + \vec{b}| = \sqrt{5^2 + (3)^2 + (-4)^2}$

$= \sqrt{25 + 9 + 16} = \sqrt{50}$

Here we get

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

By substituting the values

$(7\hat{i} + \hat{k})(5\hat{i} + 3\hat{j} - 4\hat{k}) = \sqrt{50}\sqrt{50}\cos\theta$

On further calculation

$(35 - 4) = 50 \cos \theta$

We get

$\cos \theta = 31/50$

$\theta = \cos^{-1} (31/50)$

13. If $\vec{a}$ is a unit vector such that $(\vec{x} - \vec{a}).(\vec{x} + \vec{a}) = 8$, find $|\vec{x}|$.
Solution:

It is given that $\vec{a}$ is a unit vector

We can write it as

$|\vec{a}| = 1$

We know that

$(\vec{x} - \vec{a}).(\vec{x} + \vec{a}) = 8$

So we get

$|\vec{x}|^2 - |\vec{a}|^2 = 8$

By substituting the values

$|\vec{x}|^2 = 8 + 1 = 9$

We get

$|\vec{x}| = 3$

14. Find the angles which the vector $\vec{a} = 3\hat{i} - 6\hat{j} + 2\hat{k}$ makes with the coordinate axes.
Solution:

It is given that

$\vec{a} = 3\hat{i} - 6\hat{j} + 2\hat{k}$

We can write it as

$\sqrt{a^2 + b^2 + c^2} = \sqrt{3^2 + (-6)^2 + 2^2}$

$= \sqrt{9 + 36 + 4} = \sqrt{49} = 7$

We know that

Angle with the x-axis

$\alpha = \cos^{-1}\frac{a}{\sqrt{a^2 + b^2 + c^2}}$

$= \cos^{-1}\frac{3}{7}$

Angle with the y-axis

$\beta = \cos^{-1}\frac{b}{\sqrt{a^2 + b^2 + c^2}}$

$= \cos^{-1}\frac{-6}{7}$

Angle with the z-axis

$$\gamma = \cos^{-1}\frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \cos^{-1}\frac{2}{7}$$

So the angles of the vector are

$$\cos^{-1}\frac{3}{7}, \cos^{-1}\frac{-6}{7}, \cos^{-1}\frac{2}{7}$$

15. Show that the vector $\vec{a} = (\hat{i} + \hat{j} + \hat{k})$ is equally inclined to the coordinate axes.
Solution:

It is given that

$$\vec{a} = \hat{i} + \hat{j} + \hat{k}$$

We know that

$$\sqrt{a^2 + b^2 + c^2} = \sqrt{1^2 + (1)^2 + 1^2}$$

$$= \sqrt{1 + 1 + 1} = \sqrt{3}$$

Angle with the x-axis

$$\alpha = \cos^{-1}\frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \cos^{-1}\frac{1}{\sqrt{3}}$$

Angle with the y-axis

$$\beta = \cos^{-1}\frac{b}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \cos^{-1}\frac{1}{\sqrt{3}}$$

Angle with the z-axis

$$\gamma = \cos^{-1}\frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \cos^{-1}\frac{1}{\sqrt{3}}$$

Here

$\alpha = \beta = \gamma$

Therefore, it is proved that the vector $\vec{a} = (\hat{i} + \hat{j} + \hat{k})$ is equally inclined to the coordinate axes.

16. Find a vector $\vec{a}$ of magnitude 5√2, making an angle π/4 with x-axis, π/2 with y-axis and an acute angle θ with z-axis.
Solution:

It is given that

$|\vec{a}| = 5\sqrt{2}$

Here we get

$l = \cos\alpha = \cos\pi/4 = 1/\sqrt{2}$

$m = \cos\beta = \cos\pi/2 = 0$

$n = \cos\theta$

It can be written as

$l^2 + m^2 + n^2 = 1$

By substituting the values

$$\frac{1}{\sqrt{2}}^2 + 0^2 + n^2 = 1$$

So we get

$$n^2 = 1 - \frac{1}{2} = \frac{1}{2}$$

Here

$$n = \pm\frac{1}{\sqrt{2}}$$

We know that the vector makes an acute angle with z-axis

$$n = +\frac{1}{\sqrt{2}}$$

$\vec{a} = |\vec{a}|(l\hat{i} + m\hat{j} + n\hat{k})$

By substituting the values

$\vec{a} = 5\sqrt{2}(1/\sqrt{2}\hat{i} + 1/\sqrt{2}\hat{k})$

So we get

$\vec{a} = 5(\hat{\imath} + \hat{k})$

17. Find the angle between $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$, if $\vec{a} = (2\hat{\imath} - \hat{\jmath} + 3\hat{k})$ and $\vec{b} = (3\hat{\imath} + \hat{\jmath} + 2\hat{k})$.

Solution:

It is given that

$\vec{a} = (2\hat{\imath} - \hat{\jmath} + 3\hat{k})$

$\vec{b} = (3\hat{\imath} + \hat{\jmath} + 2\hat{k})$

We know that

$\vec{a} + \vec{b} = (2\hat{\imath} - \hat{\jmath} + 3\hat{k}) + (3\hat{\imath} + \hat{\jmath} + 2\hat{k})$

$= 5\hat{\imath} + 5\hat{k}$

$\vec{a} - \vec{b} = (2\hat{\imath} - \hat{\jmath} + 3\hat{k}) - (3\hat{\imath} + \hat{\jmath} + 2\hat{k})$

$= -\hat{\imath} - 2\hat{\jmath} + \hat{k}$

So we get

$|\vec{a} + \vec{b}| = \sqrt{5^2 + (5)^2}$

$= \sqrt{25 + 25} = \sqrt{50}$

$|\vec{a} - \vec{b}| = \sqrt{(-1)^2 + (-2)^2 + (1)^2}$

$= \sqrt{1 + 4 + 1} = \sqrt{6}$

Here

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

By substituting the values

$(5\hat{\imath} + 5\hat{k})(-\hat{\imath} - 2\hat{\jmath} + \hat{k}) = \sqrt{50}\sqrt{6}\cos\theta$

On further calculation

$(-5 + 5) = \sqrt{300} \cos \theta$

We get

$\cos\theta = 0$
$\theta = \cos^{-1}(0) = \pi/2$